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TWO-DIMENSIONAL, POWER-LAW WINGS OF MAXIMUM LIFT-TO-DRAG RATIO IN HYPERSONIC FLOW

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ANGELO MIELE and WILLIAM L. WILSON

RICE UNIVERSITY

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SUMMARY

The problem of maximizing the lift-to-drag ratio of a slender, flat-top, hypersonic wing is investigated under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is constant. Direct methods are employed, and the analysis is confined to the class of two-dimensional wings whose chordwise contour is a power law.

over →

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(**) Professor of Astronautics and Director of the Aero-Astronautics Group, Department of Mechanical and Aerospace Engineering and Materials Science, Rice University, Houston, Texas.

(***) Graduate Student in Astronautics, Department of Mechanical and Aerospace Engineering and Materials Science, Rice University, Houston, Texas.

First, unconstrained configurations are considered, and the combination of power law exponent and thickness ratio maximizing the lift-to-drag ratio is determined. For a friction coefficient $C_f = 10^{-3}$, the maximum lift-to-drag ratio is E = 5.29 and corresponds to a wedge of thickness ratio $\tau = 0.126$.

Next, constrained configurations are considered, that is, conditions are imposed on the length, the thickness, the enclosed area, and the position of the center of pressure. For each combination of constraints, an appropriate similarity parameter is introduced, and the optimum power law exponent, thickness ratio, and lift-to-drag ratio are determined as functions of the similarity parameter.

AUTHOR

1. INTRODUCTION

In Ref. 1, the basic theory of slender, flat-top, affine wings in the hypersonic regime was formulated under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is constant. Analytical expressions were derived relating the drag, the lift, and the lift-to-drag ratio to the geometry of the configuration, that is, the chordwise and spanwise contours of the affine wing. In Ref. 2, two complementary variational problems were formulated, that of optimizing the chordwise contour and that of optimizing the spanwise contour and the chord distribution, the criterion of optimization being the lift-to-drag ratio. The existence of similar solutions was investigated, and it was concluded that (1) the optimum chordwise contour of a wing of arbitrary spanwise contour and chord distribution can be determined from the known optimum chordwise contour of a wing of constant trailing edge thickness and constant chord and (2) the optimum spanwise contour and chord distribution of a wing of arbitrary chordwise contour can be determined from the known optimum spanwise contour and chord distribution of a wing of linear chordwise thickness distribution.

The next step is to determine the extremal properties of these reference wings. Here, a two-dimensional wing is considered, and its chordwise contour is determined for given constraints imposed on the length, the thickness, the enclosed area, and the position of the center of pressure. Direct methods are employed, and the analysis is confined to the class of power law contours. In a subsequent report (Ref. 3), this limitation is removed, and the chordwise contour is determined with the indirect methods of the calculus of variations.

The hypotheses employed are as follows: (a) the wing is two-dimensional; (b) the upper surface is flat; (c) the free-stream velocity is parallel to the plane of the flat top and is perpendicular to the base plane; (d) the pressure coefficient is twice the cosine squared of the angle formed by the free-stream velocity and the normal to each surface element; (e) the skin-friction coefficient is constant; (f) the contribution of the tangential forces to the lift is negligible with respect to the contribution of the normal forces; (g) the wing is slender in the chordwise sense; and (h) the chordwise contour is represented by a power law.

2. FUNDAMENTAL EQUATIONS

In order to relate the drag, the lift, and the pitching moment of a two-dimensional wing to its geometry, we define the following Cartesian coordinate system Oxz (Fig. 1): the origin O is the leading edge; the x-axis is identical with the free-stream direction and positive toward the trailing edge; and the z-axis is perpendicular to the x-axis and positive downward.

If the hypotheses (a) through (g) are invoked and if the lower surface is represented by the relationship

$$z = z(x) \tag{1}$$

the drag D, the lift L, and the pitching moment M per unit span can be written as (Ref. 1 and 2)

$$\begin{aligned} D/q &= 2 \int_0^{\ell} (\dot{z}^3 + C_f)\, dx \\ L/q &= 2 \int_0^{\ell} \dot{z}^2\, dx \\ M/q &= 2 \int_0^{\ell} x\dot{z}^2\, dx \end{aligned} \tag{2}$$

where q is the free-stream dynamic pressure and $\dot{z}$ the derivative dz/dx. In accordance with hypothesis (h),we specialize the chordwise contour (1) to the power law

$$z/t = (x/\ell)^n \tag{3}$$

in which n is an undetermined exponent and t is the base thickness. Consequently,

Eqs.(2) become

$$\begin{aligned} D/q &= \ell\left(\tau^3 f_1 + 2C_f\right) \\ L/q &= \ell\tau^2 f_2 \\ M/q &= \ell^2 \tau^2 f_3 \end{aligned} \tag{4}$$

where

$$\tau = t/\ell \tag{5}$$

is the thickness ratio and where

$$\begin{aligned} f_1(n) &= 2n^3/(3n - 2) \\ f_2(n) &= 2n^2/(2n - 1) \\ f_3(n) &= n \end{aligned} \tag{6}$$

These equations are valid for $n > 2/3$ only, owing to the fact that the pressure drag cannot be negative.

Once the drag, the lift, and the pitching moment are known, certain derived quantities can be calculated. They are the lift-to-drag ratio E and the nondimensional distance ξ_o of the center of pressure from the apex. These quantities are defined by

$$E = L/D \qquad \xi_o = x_o/\ell = M/L\ell \tag{7}$$

and, because of Eqs. (4), can be rewritten as

$$E = \tau^2 f_2/(\tau^3 f_1 + 2C_f) \quad , \quad \xi_o = f_3/f_2 \tag{8}$$

Clearly, the lift-to-drag ratio depends on both the thickness ratio and the power law exponent, while the position of the center of pressure depends on the power law exponent only.

Finally, the area enclosed by the profile of a flat-top, two-dimensional wing is given by

$$A = \int_0^\ell z dx \tag{9}$$

and reduces to

$$A = \ell^2 \tau f_4 \tag{10}$$

with

$$f_4(n) = 1/(n + 1) \tag{11}$$

if the chordwise contour is represented by a power law.

3. UNCONSTRAINED CONFIGURATION

The first step in the analysis is to determine the maximum lift-to-drag ratio of a configuration which is unconstrained geometrically and aerodynamically. According to Eq. (8-1), the lift-to-drag ratio depends on both the thickness ratio and the power law exponent, that is, it has the form $E = E(\tau, n)$. Therefore, the optimum values of τ and n are determined by the simultaneous relationships

$$E_{\tau} = 0 \quad , \quad E_{n} = 0 \tag{12}$$

in which the subscripts denote partial derivatives. These relationships can be written explicitly as

$$\begin{aligned} &\tau^3 f_1 - 4C_f = 0 \\ &\dot{f}_2(\tau^3 f_1 + 2C_f) - \tau^3 f_2 \dot{f}_1 = 0 \end{aligned} \tag{13}$$

with the dot sign denoting a derivative with respect to n. From Eq. (13-1), it appears that the optimum thickness ratio is such that the skin-friction drag is one-third of the total drag. Furthermore, upon eliminating the thickness ratio from Eqs. (13), we obtain the relationship

$$2g_1 - 3g_2 = 0 \tag{14}$$

where

$$g_1 = \dot{f}_1/f_1 \quad , \quad g_2 = \dot{f}_2/f_2 \tag{15}$$

On account of the definitions (6-1) and (6-2), we see that Eq. (14) is solved by

$$n = 1 \tag{16}$$

which means that the optimum chordwise contour is that of a wedge. With this understanding, the thickness ratio (13-1) and the lift-to-drag ratio (8-1) become

$$\begin{aligned} \tau C_f^{-1/3} &= \sqrt[3]{2} \cong 1.26 \\ E C_f^{1/3} &= 2/3 \sqrt[3]{2} \cong 0.529 \end{aligned} \tag{17}$$

Equation (17-2) represents the upper limit to the lift-to-drag ratio which can be obtained with a two-dimensional, flat-top configuration subjected to a flow parallel to the flat top. Should the configuration be required to satisfy a certain number of geometric and/or aerodynamic constraints, a loss in the lift-to-drag ratio would occur with respect to that predicted by Eq. (17-2).

4. GIVEN CENTER OF PRESSURE

To prescribe the nondimensional distance of the center of pressure from the apex is equivalent to prescribing the power law exponent in accordance with Eq. (8-2). Therefore, the lift-to-drag ratio can be maximized with respect to the thickness ratio only, and the relevant optimum conditions is represented by Eq. (12-1) implicitly or Eq. (13-1) explicitly. Because of Eq. (13-1), the optimum thickness ratio is given by

$$\tau C_f^{-1/3} = (4/f_1)^{1/3} \tag{18}$$

and the associated lift-to-drag ratio is

$$E C_f^{1/3} = (f_2/3)(2/f_1^2)^{1/3} \tag{19}$$

The parametric equations (8-2), (18), and (19) admit solutions of the form

$$n = P(\xi_o) \quad , \quad \tau C_f^{-1/3} = Q(\xi_o) \quad , \quad E C_f^{1/3} = R(\xi_o) \tag{20}$$

which are plotted in Figs. 2 through 4. For $\xi_o = 1/2$, the chordwise contour is that of a wedge, and the maximum lift-to-drag ratio reaches the upper limit represented by Eq. (17-2). For any other value of ξ_o, lower values of the lift-to-drag ratio are obtained as shown in Fig. 4.

5. GIVEN THICKNESS AND LENGTH

To prescribe the thickness and the length is equivalent to prescribing the thickness ratio τ in accordance with the definition (5). Therefore, the lift-to-drag ratio (8-1) can be maximized with respect to the power law exponent only, and the relevant optimum condition is represented by Eq. (12-2) implicitly or Eq. (13-2) explicitly. Because of Eq. (13-2), the optimum power law exponent satisfies the relationship

$$\tau C_f^{-1/3} = \left(\frac{2}{g_1 - g_2}\right)^{1/3} \left(\frac{f_1}{g_2}\right)^{-1/3} \tag{21}$$

The associated lift-to-drag ratio is given by

$$E C_f^{1/3} = \left(\frac{2}{g_1 - g_2}\right)^{-1/3} \left(\frac{f_1}{g_2}\right)^{-2/3} \left(\frac{f_2}{g_1}\right) \tag{22}$$

The parametric equations (21) and (22) admit solutions of the form(*)

$$n = P(\tau C_f^{-1/3}) \quad , \quad E C_f^{1/3} = R(\tau C_f^{-1/3}) \tag{23}$$

(*) The functions (23) are triple-valued for $\tau C_f^{-1/3} \leq 0.830$ and single-valued for $\tau C_f^{-1/3} \geq 0.830$. In the former case, there is one relative minimum solution and two relative maximum solutions. Among the latter, one must determine the absolute maximum solution by direct comparison of the lift-to-drag ratio. This maximum solution is plotted in Figs. 5 and 6.

which are plotted in Figs. 5 and 6. If the thickness-length parameter is smaller than 0.825, the configuration is concave. If the thickness-length parameter is larger than 0.825, the configuration is a wedge which, for $\tau C_f^{-1/3} = 1.26$, has the lift-to-drag ratio $E C_f^{1/3} = 0.529$.

6. GIVEN ENCLOSED AREA AND LENGTH

If the enclosed area and the length are given, it is convenient to rewrite Eq. (10) in the form

$$A/\ell^2 = \tau f_4 \tag{24}$$

The lift-to-drag ratio (8-1) is to be maximized with respect to the combinations of τ and n which ensure the constancy of the right-hand side of Eq. (24). In accordance with Lagrange multiplier theory, we introduce an undetermined constant λ and define the fundamental function

$$F = E + \lambda\tau f_4 \tag{25}$$

Then, the optimum conditions are

$$F_\tau = 0 \quad , \quad F_n = 0 \tag{26}$$

which are equivalent to

$$E_\tau + \lambda f_4 = 0 \quad , \quad E_n + \lambda\tau \dot{f}_4 = 0 \tag{27}$$

and, upon elimination of the Lagrange multiplier, imply that

$$\tau g_4 E_\tau - E_n = 0 \tag{28}$$

where

$$g_4 = \dot{f}_4/f_4 \tag{29}$$

In the light of Eq. (8-1), Eq. (28) can be rewritten as

$$\tau C_f^{-1/3} = \left(\frac{2}{g_1 - g_2 - g_4}\right)^{1/3} \left(\frac{f_1}{g_2 - 2g_4}\right)^{-1/3} \tag{30}$$

The associated lift-to-drag ratio and area-length parameter are given by

$$E C_f^{1/3} = \left(\frac{2}{g_1 - g_2 - g_4}\right)^{-1/3} \left(\frac{f_1}{g_2 - 2g_4}\right)^{-2/3} \left(\frac{f_2}{g_1 - 3g_4}\right)$$

$$A\ell^{-2} C_f^{-1/3} = \left(\frac{2}{g_1 - g_2 - g_4}\right)^{1/3} \left(\frac{f_1}{g_2 - 2g_4}\right)^{-1/3} f_4 \tag{31}$$

The parametric equations (30) and (31) admit solutions of the form(*)

$$n = P(A\ell^{-2} C_f^{-1/3}) \;, \quad \tau C_f^{-1/3} = Q(A\ell^{-2} C_f^{-1/3}) \;, \quad E C_f^{1/3} = R(A\ell^{-2} C_f^{-1/3}) \tag{32}$$

which are plotted in Figs. 7 through 9. When the area-length parameter has the value

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(*) The functions (32) are triple-valued for $A\ell^{-2} C_f^{-1/3} \leq 0.169$ and single-valued for $A\ell^{-2} C_f^{-1/3} \geq 0.169$. In the former case, there is one relative minimum solution and two relative maximum solutions. Among the latter, one must determine the absolute maximum solution by direct comparison of the lift-to-drag ratio. This maximum solution is plotted in Figs. 7 through 9.

0.630, the configuration is a wedge with a thickness ratio $\tau C_f^{-1/3} = 1.26$ and a lift-to-drag ratio $EC_f^{1/3} = 0.529$. For larger values of the area-length parameter, the configuration is convex, and for smaller values, it is concave.

7. GIVEN ENCLOSED AREA AND THICKNESS

If the enclosed area and the thickness are given, it is convenient to rewrite Eq. (10) in the form

$$A/t^2 = f_4/\tau \tag{33}$$

The lift-to-drag ratio (8-1) is to be maximized with respect to the combinations of τ and n which ensure the constancy of the right-hand side of Eq. (33). In accordance with Lagrange multiplier theory, we introduce an undetermined constant λ and define the fundamental function

$$F = E + \lambda f_4/\tau \tag{34}$$

Then, the optimum conditions are

$$F_\tau = 0 \quad , \quad F_n = 0 \tag{35}$$

which are equivalent to

$$\tau^2 E_\tau - \lambda f_4 = 0 \quad , \quad \tau E_n + \lambda \dot{f}_4 = 0 \tag{36}$$

and, upon elimination of the Lagrange multiplier, imply that

$$\tau g_4 E_\tau + E_n = 0 \tag{37}$$

In the light of Eq. (8-1), Eq. (37) can be rewritten as

$$\tau C_f^{-1/3} = \left(\frac{2}{g_1 - g_2 + g_4}\right)^{1/3} \left(\frac{f_1}{g_2 + 2g_4}\right)^{-1/3} \tag{38}$$

The associated lift-to-drag ratio and area-thickness parameter are given by

$$\begin{aligned} EC_f^{1/3} &= \left(\frac{2}{g_1 - g_2 + g_4}\right)^{-1/3} \left(\frac{f_1}{g_2 + 2g_4}\right)^{-2/3} \left(\frac{f_2}{g_1 + 3g_4}\right) \\ At^{-2}C_f^{1/3} &= \left(\frac{2}{g_1 - g_2 + g_4}\right)^{-1/3} \left(\frac{f_1}{g_2 + 2g_4}\right)^{1/3} f_4 \end{aligned} \tag{39}$$

The parametric equations (38) and (39) admit solutions of the form

$$n = P(At^{-2}C_f^{1/3}) \quad , \quad \tau C_f^{-1/3} = Q(At^{-2}C_f^{1/3}) \ , \ EC_f^{1/3} = R(At^{-2}C_f^{1/3}) \tag{40}$$

which are plotted in Figs. 10 through 12. When the area-thickness parameter has the value 0.397, the configuration is a wedge, with a thickness ratio $\tau C_f^{-1/3} = 1.26$ and a lift-to-drag ratio $EC_f^{1/3} = 0.529$. For larger values of the area-thickness parameter, the configuration is convex, and for smaller values, it is concave.

8. DISCUSSION AND CONCLUSIONS

In the previous sections, the problem of maximizing the lift-to-drag ratio of a slender, flat-top, hypersonic wing is investigated under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is constant. Direct methods are employed, and the analysis is confined to the class of two-dimensional wings whose chordwise contour is a power law.

First, unconstrained configurations are considered, and the combination of power law exponent and thickness ratio maximizing the lift-to-drag ratio is determined. For a friction coefficient $C_f = 10^{-3}$, the maximum lift-to-drag ratio is $E = 5.29$ and corresponds to a wedge of thickness ratio $\tau = 0.126$.

Next, constrained configurations are considered, that is, given conditions are imposed on the length, the thickness, the enclosed area, and the position of the center of pressure. For each combination of constraints, an appropriate similarity parameter is introduced, and the optimum power law exponent, thickness ratio, and lift-to-drag ratio are determined as functions of the similarity parameter. The lift-to-drag ratio of a constrained configuration is smaller than that of the optimum unconstrained configuration; however, for a particular value of the similarity parameter, equality is achieved.

While the chordwise contour is that of a wedge for an unconstrained configuration, it can be either convex or concave for constrained configurations, depending on the value of the similarity parameter. Since the Newtonian pressure law has been verified experimentally for convex configurations only, the results pertaining to concave configurations are merely indicative of qualitative trends.

Finally, it is of interest to compare the present lift-to-drag ratios with those characteristic of drag-optimized, flat-top configurations. The analysis is omitted for the sake of brevity,since it involves only a slight modification of that presented in Ref. 4. As expected, the lift-to-drag ratio of a minimum drag configuration is lower than that of a maximum lift-to-drag ratio configuration (see Figs. 5 through 12).

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LIST OF CAPTIONS

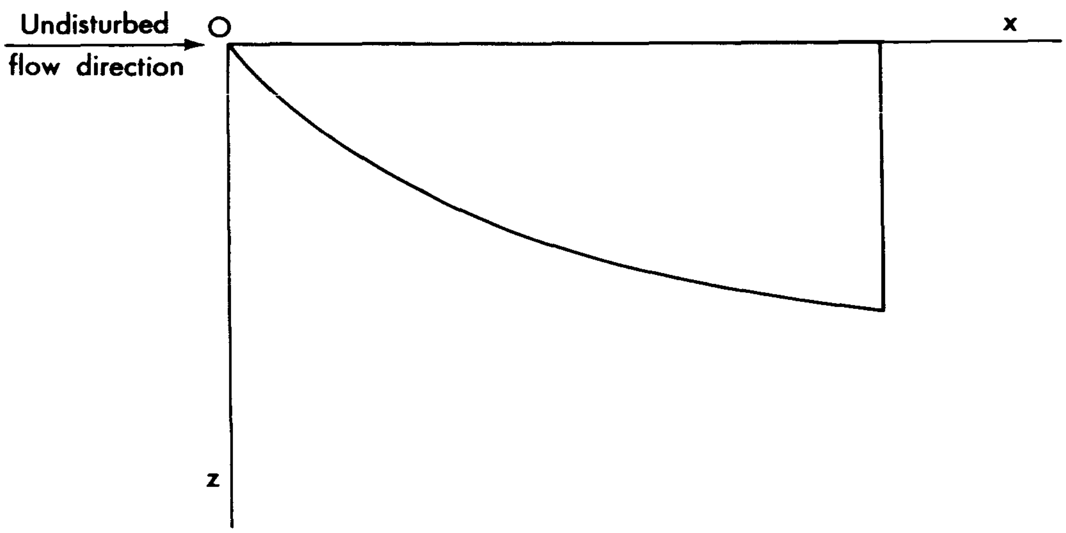

Fig. 1 Coordinate system.

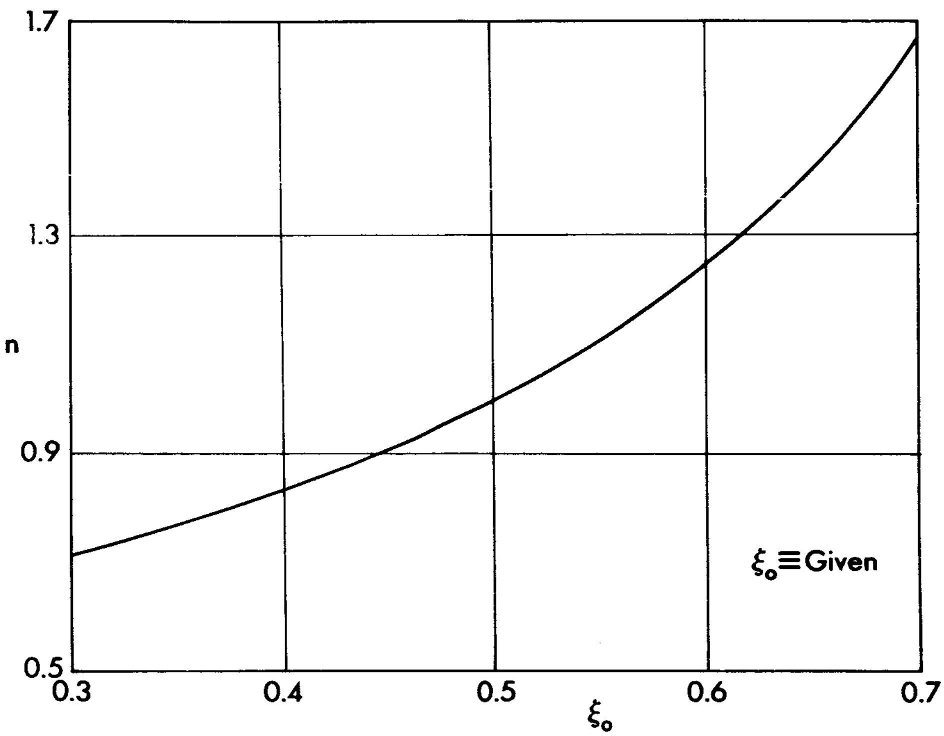

Fig. 2 Power law exponent.

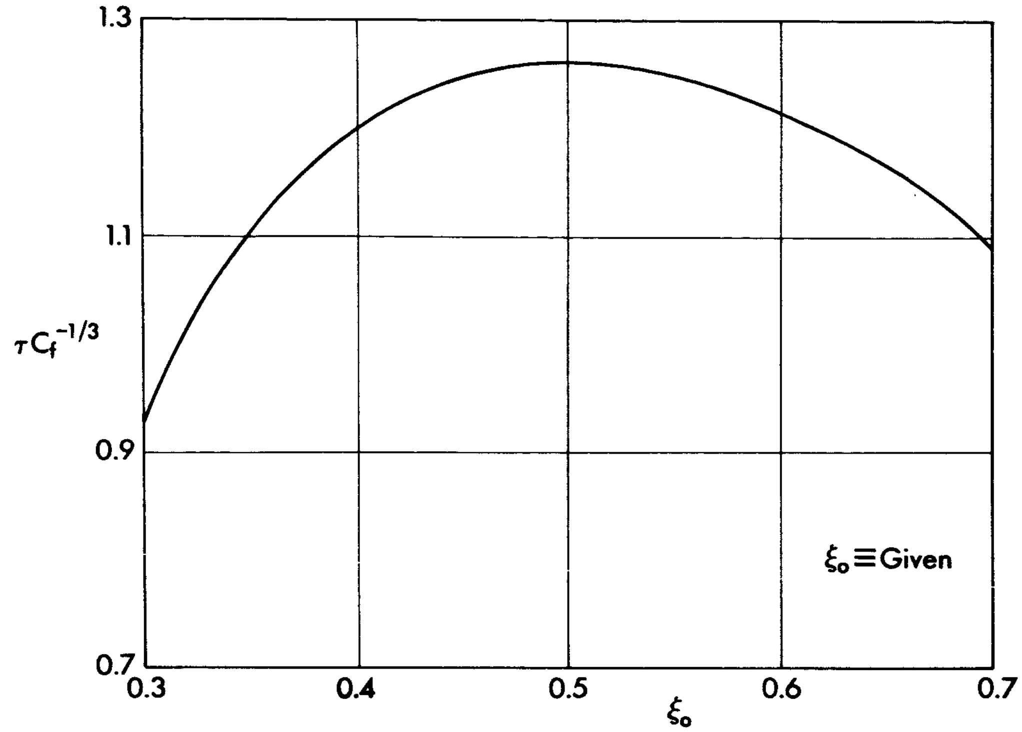

Fig. 3 Optimum thickness ratio.

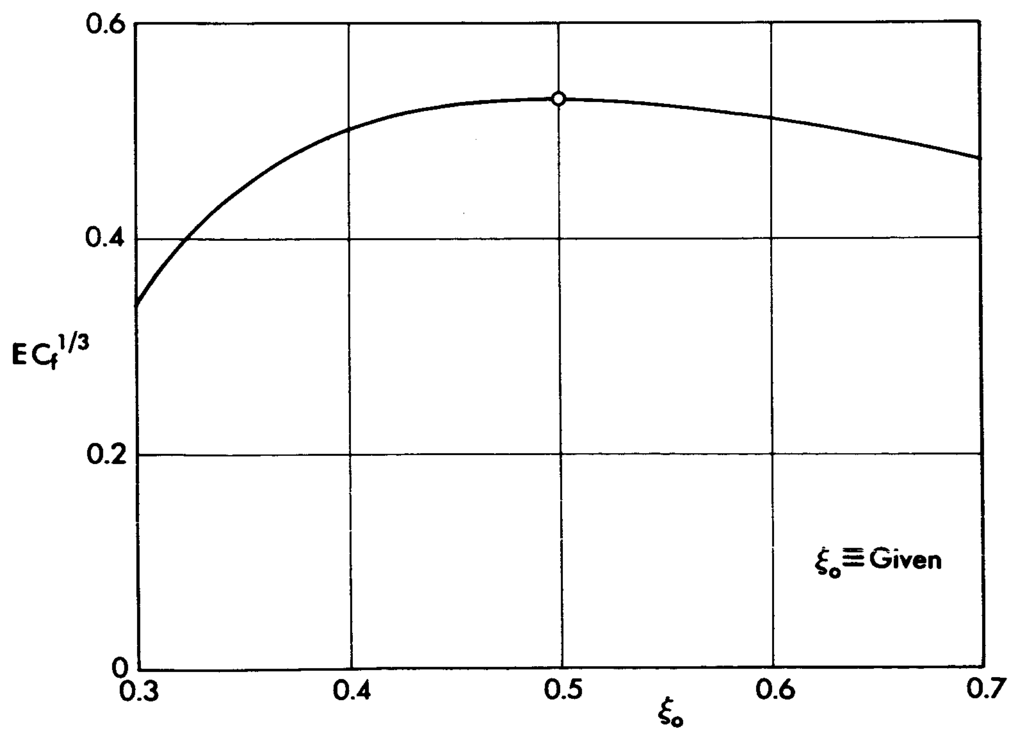

Fig. 4 Maximum lift-to-drag ratio.

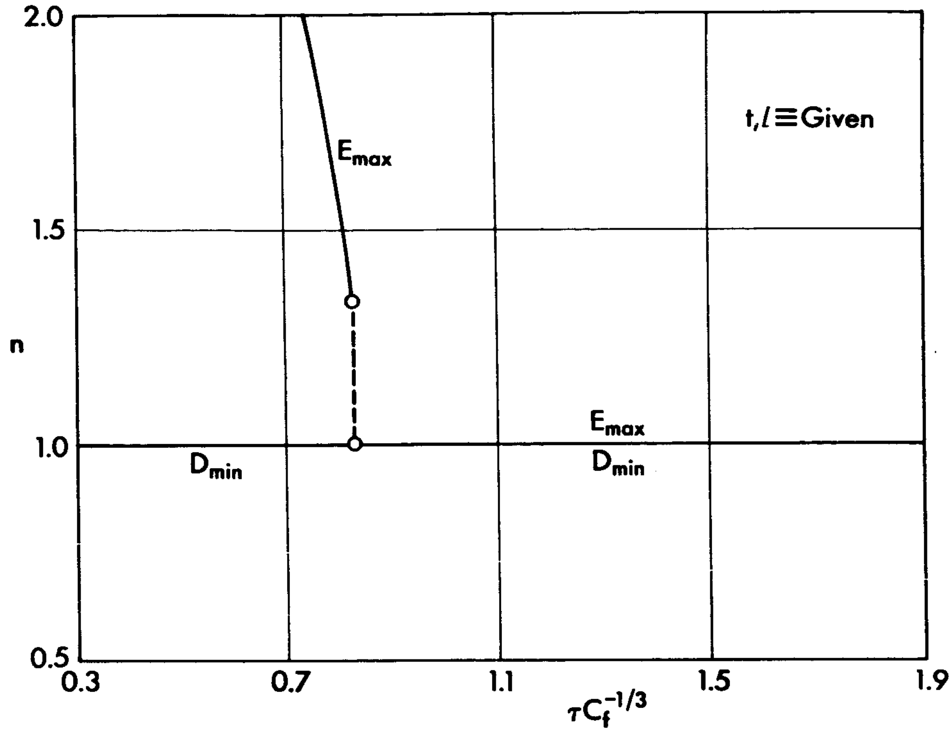

Fig. 5 Optimum power law exponent.

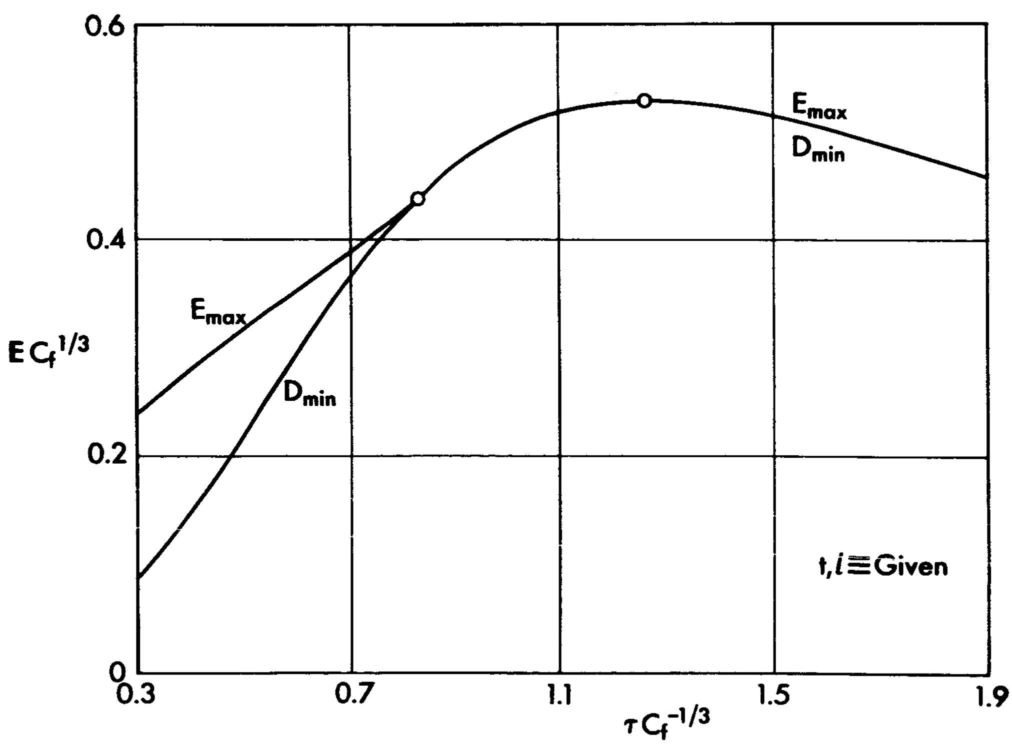

Fig. 6 Maximum lift-to-drag ratio.

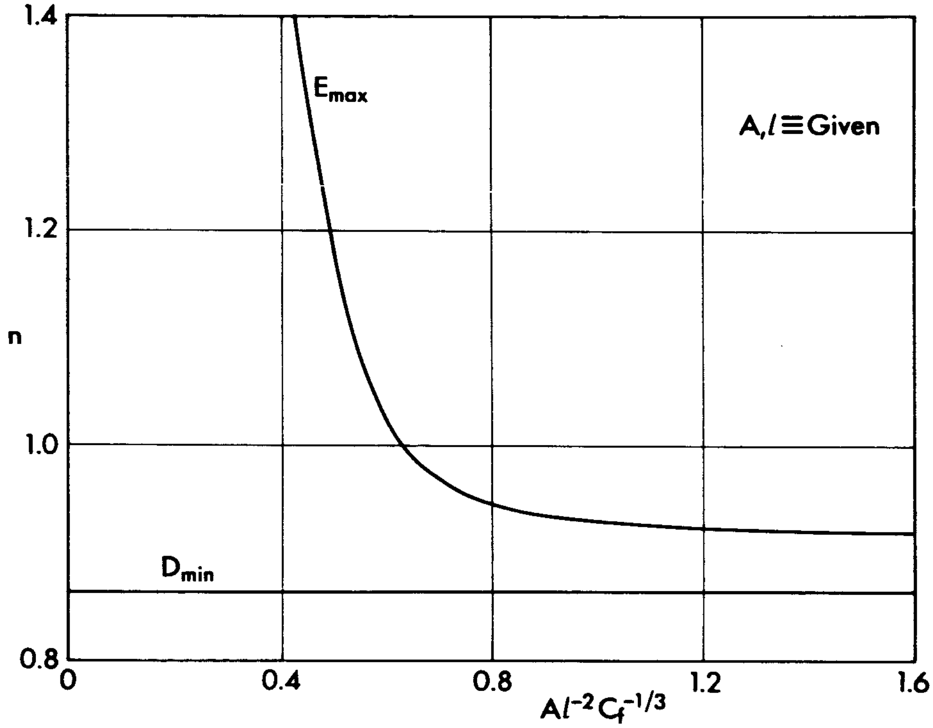

Fig. 7 Optimum power law exponent.

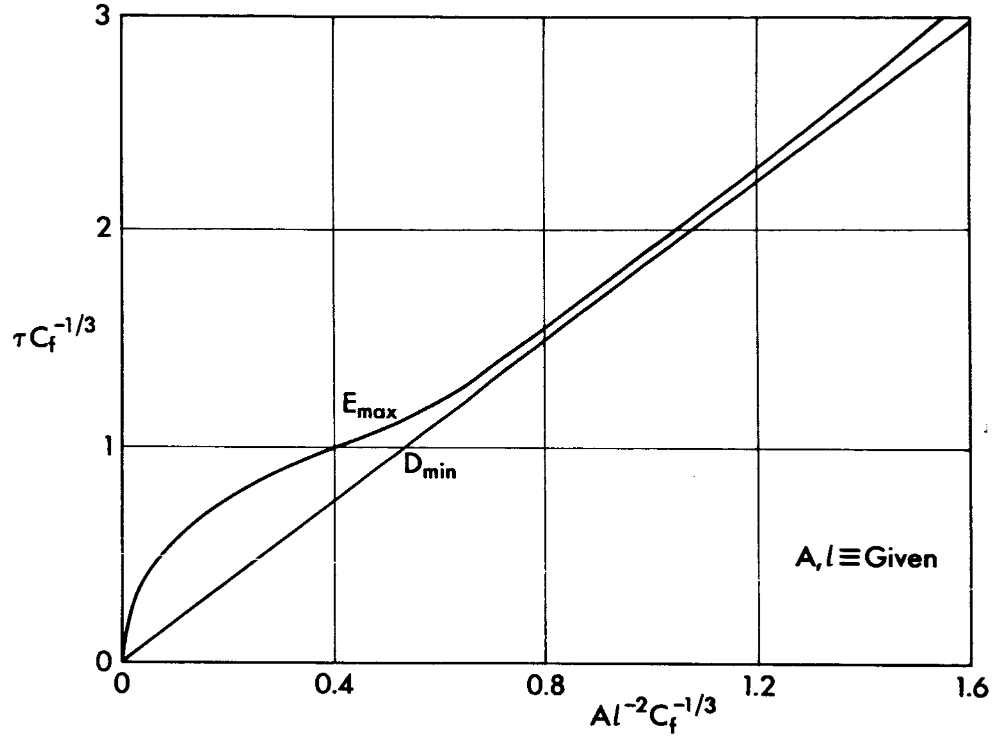

Fig. 8 Optimum thickness ratio.

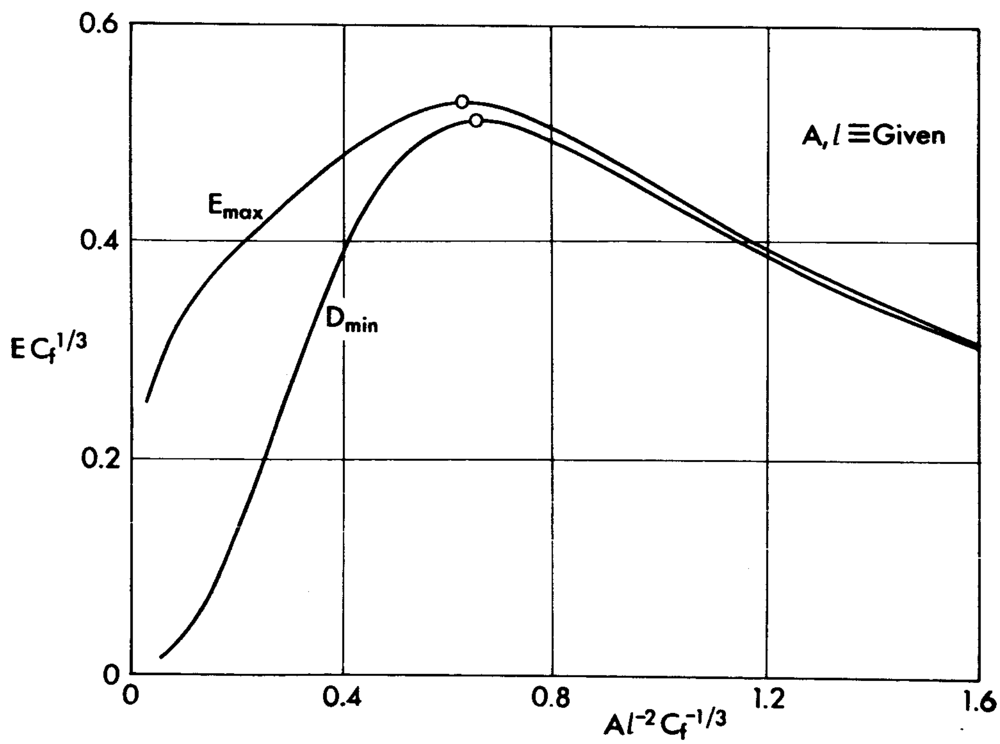

Fig. 9 Maximum lift-to-drag ratio.

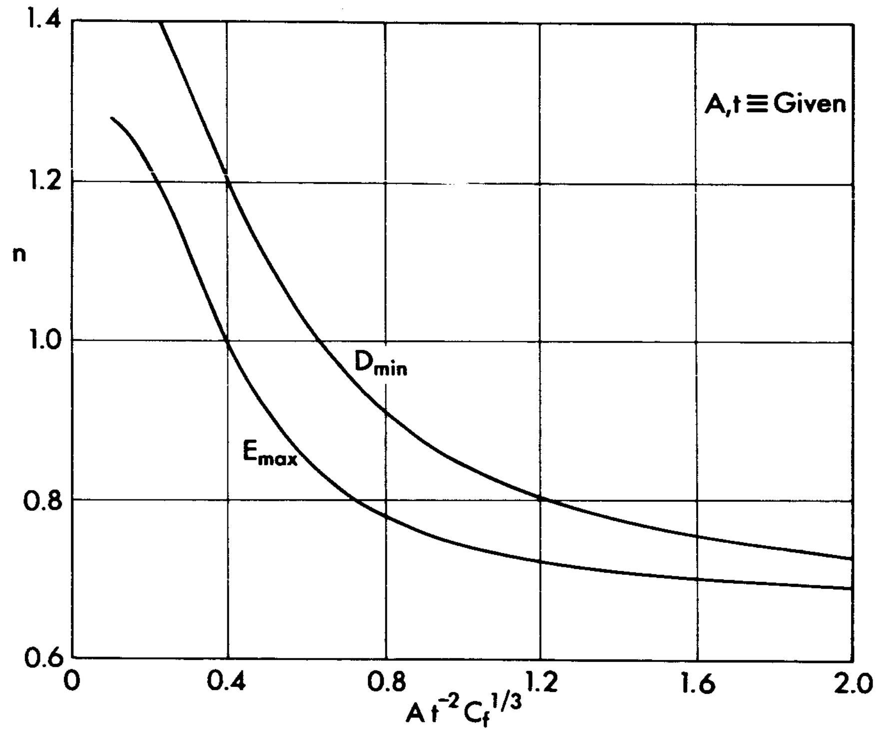

Fig. 10 Optimum power law exponent.

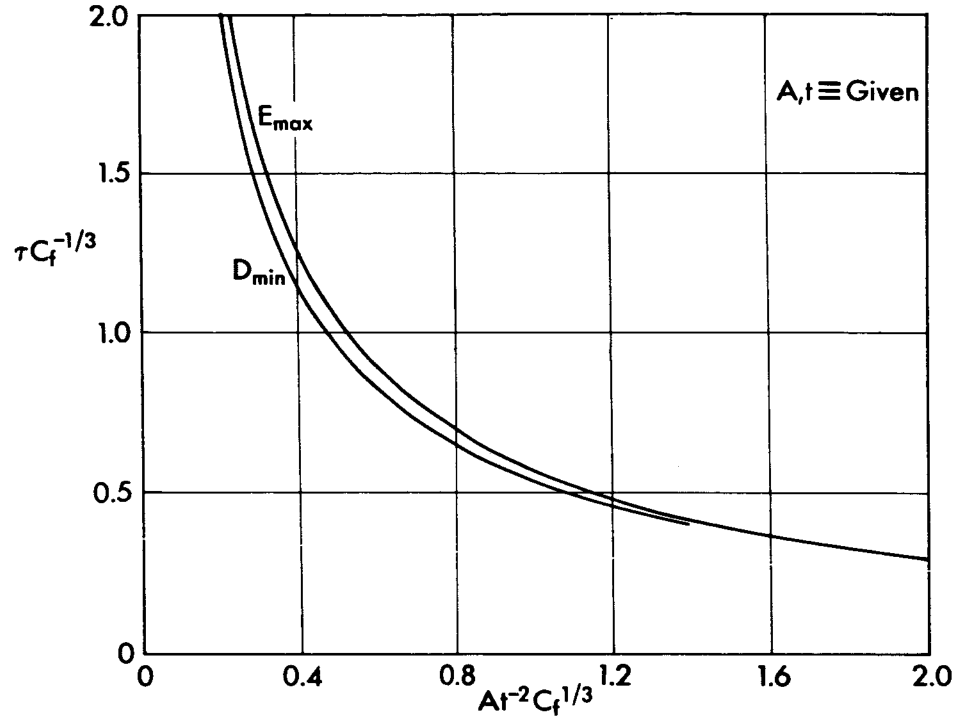

Fig. 11 Optimum thickness ratio.

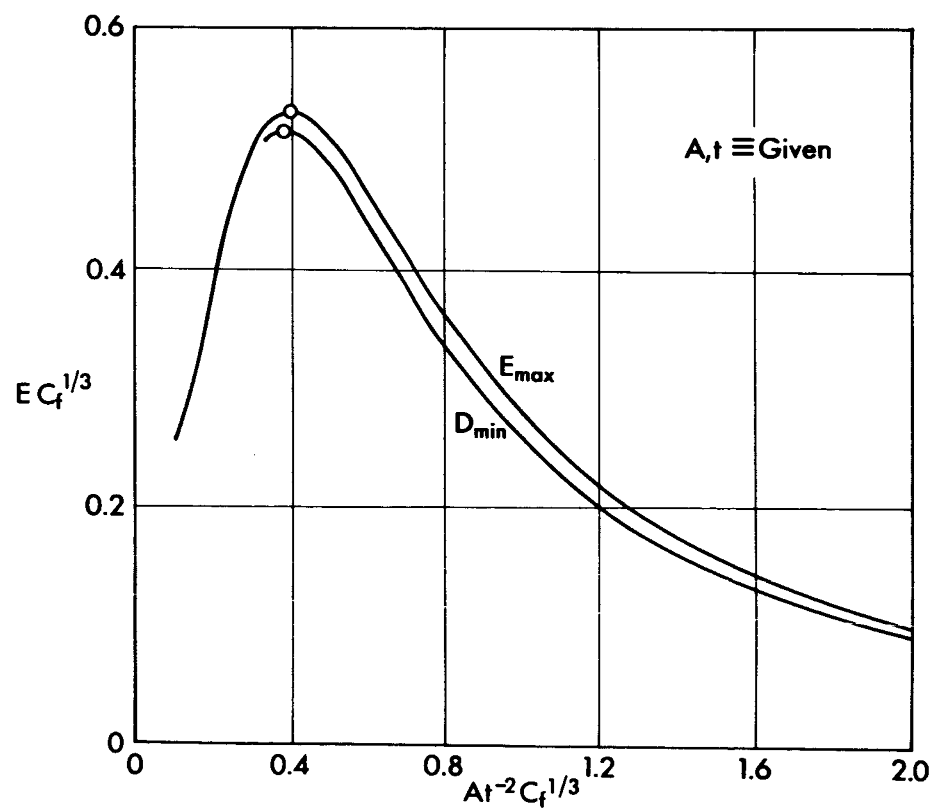

Fig. 12 Maximum lift-to-drag ratio.